Electrical Engineering 229A Lecture 20 Notes

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1 Capacity of Wide Sense Stationary Processes and Parallel Gaussian channels

1.1 Wide sense stationary processes

Definition 1.1. A **stationary stochastic process** $(X(t).t \in \mathbb{R})$ is a collection of random variables $X(t)$ such that

$$(X(t_1), \dots, X(t_d)) \stackrel{d}{=} (X(t_1 + s), \dots, X(t_d + s)).$$

The correct thing to study to understand spectral properties of such a process is the autocorrelation function.

Definition 1.2. The **autocorrelation function** is

$$R_{x,x}(s,t) := \mathbb{E}[X(t)X(s)] = R_x(t-s).$$

By stationarity, this only depends on $t-s$.

Definition 1.3. A **wide sense stationary (WSS) process** is a process for which $R_{x,x}(t,s)$ depends only on $t-s$ (and $\mathbb{E}[X(t)]$ is constant).

Definition 1.4. The **power spectral density** of the noise is

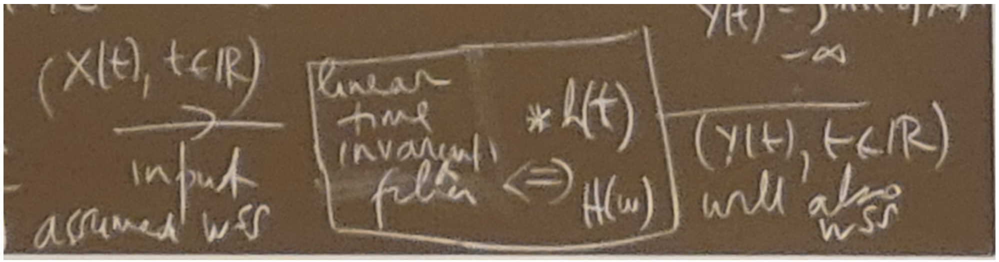
$$S_{x,x}(\omega) = \int_{-\infty}^{\infty} R_{x,x}(t) e^{-i\omega\tau}\, d\tau,$$

the Fourier transform of the autocorrelation function.[1]

If we input a WSS into a linear time invariant filter, which outputs a WSS, then we have the following magic formula:

$$S_{y,y}(\omega) = |H(\omega)|^2 S_{x,x}(\omega).$$

[1]Professor Anantharam uses j instead of i, but I disagree.



We should think of $S_{x,x}$ as telling us how much noise sits at each frequency.

Definition 1.5. If $S_{x,x}(\omega)$ is constant, then $(X(t), t \in \mathbb{R})$ is called **white noise**. If in addition, $(X(t), t \in \mathbb{R})$ is a Gaussian process, i.e. $(X(t_1), \dots, X(t_d))$ is jointly Gaussian for all $t_1, \dots, t_d$, we call this **white Gaussian noise**.

Assuming $\mathbb{E}[X(t)] = 0$ for all t, this is characterized by the properties

1. $$\int_{-\infty}^{\infty} X(t) f(t)\, dt \sim N(0, \sigma^2), \quad \text{if } \int_{-\infty}^{\infty} f^2(t)\, dt,$$

2. $$\int_{-\infty}^{\infty} f(t) g(t)\, dt = 0 \implies \int_{-\infty}^{\infty} X(t) f(t)\, dt \perp\!\!\!\perp \int_{-\infty}^{\infty} X(t) g(t)\, dt.$$

1.2 Connection between WSSs and AWGNs

Last time, we saw that the Shannon capacity of a Power-constrained AWGN is

$$\frac{1}{2} \log \left(1 + \frac{P}{\sigma^2}\right) \qquad \text{bits per use.}$$

This is interesting because it is a model for if you input a power-constrained waveform X (bandlimited to W Hz and time limited to T seconds) and the noise Z is additive and white Gaussian noise. Here, the output is $Y(t) = X(t) + Z(t)$.

The number of degrees of freedom, which represents the dimension of our input, is intuitively $2WT$. Nyquist sampling theory tells us that $2W$ samples per second is needed to recover a signal which is bandlimited to W. The Landau-Pollack paper makes this precise via prolate spheroidal functions.

The functions for which a fraction of at least $1 - \varepsilon_T^2$ of the entropy should be on $[-T/2, T/2]$ and which are bandlimited to W can be expressed in terms of $2WT +$constant prolate spheroidal functions, capturing at least $1 - c\varepsilon_T^2$ of the energy. Here, $\varepsilon_T \to 0$ as $T \to \infty$.

The number of uses of the AWGN is replaced by $2WT$, and the power on a per use basis is replaced by power on a per degree of freedom basis. Let P denote power on a per unit time basis; then the power on a per degree of freedom basis is $\frac{P}{2W}$. The noise power σ^2 on a per use basis is replaced by the noise power per degree of freedom, $\frac{N_0}{2}$. The formula we get is

$$\frac{1}{T}\left(2WT\frac{1}{2}\log\left(1+\frac{P/(2W)}{(N_0/2)}\right)\right) = W\log\left(1+\frac{P}{N_0W}\right) \quad \text{bits per unit time.}$$

Remark 1.1. Here is a practically important observation for space communication: For fixed P,

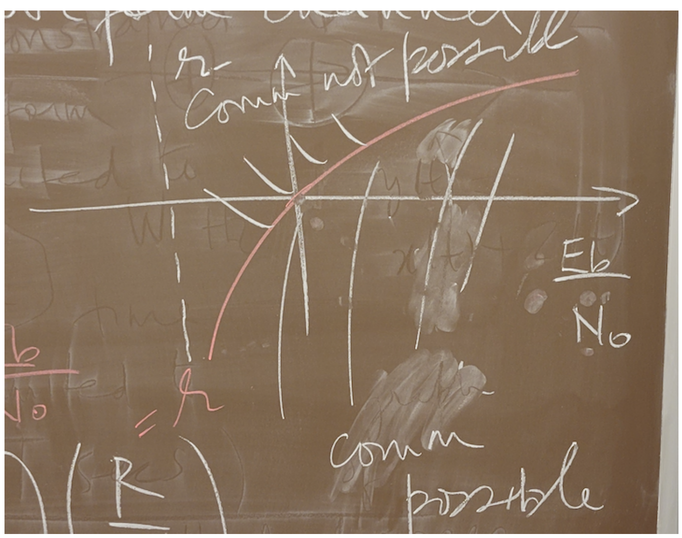
$$\lim_{W\to\infty} W\log\left(1+\frac{P}{N_0W}\right) = \frac{P}{N_0}\log_2 e \approx 1.44\frac{P}{N_0} \quad \text{bits per second.}$$

So even with infinite bandwidth, the communication rate is power-limited.

In situations where bandwidth is limited (e.g. terrestrial communication), we call R/W (denoted r) is called the **spectral efficiency** (bits/sec per Hz), and $P/(N_0R)$ (denoted E_b/N_0) is called the **signal to noise per bit**; here R is the rate of communication. Shannon's theorem for the white Gaussian noise waveform channel can be written as saying: We must have

$$r < \log\left(1+\frac{E_b}{N_0}r\right).$$

This is considered a very insightful restatement of $R < W\log(1+\frac{P}{N_0W})$. Here is a graph (in a log-log scale) of the region in which communication is possible:



What is astonishing is that you need at least a minimum value of E_b/N_0 to communicate at all!

1.3 The Shannon capacity of a parallel Gaussian channel

Leading up to the waveform channel Shannon capacity over colored noise, we'll first study the **parallel Gaussian channel** model. At each channel use, say at time i, we have a vector of inputs $(X_i^{(1)}, \ldots, X_i^{(K)})$, each of which has some added independent Gaussian noise $Z_i^{(k)}$. We receive a vector of outputs $(Y_i^{(1)}, \ldots, Y_i^{(K)})$. Here, $Z_i^{(k)} \sim \mathcal{N}(0, \sigma_k^2)$ are independent over i and k for $k = 1, \ldots, K$ and $i = 1, 2, \ldots$.

When coding at block-length n, we require for each message $m \in [M_n]$ that

$$\sum_{i=1}^{n} \sum_{k=1}^{K} (x_i^{(k)}(m))^2 \leq nP.$$

where the term in the sum is the total energy in the codeword for message m.

Theorem 1.1. *In the parallel Gaussian channel model, the Shannon capacity is*

$$\sup_{\sum_{k=1}^{K} \mathbb{E}[(X^{(k)})^2] \leq P} I(X^{(1)}, \ldots, X^{(K)}; Y^{(1)}, \ldots, Y^{(K)})$$

We will discuss this further next time.